Two-loop Correction to the Instanton Density for the Double Well Potential

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Abstract

Feynman diagrams in the instanton background are used for the calculation of the tunneling amplitude, up to the two-loops order. Some mistakes made in the previous works are corrected. The same method is applied to the next-order corrections to the ground state wave function.

A motion in the double-well potential is a classic example, displaying tunneling phenomena in their simplest form. its discussion was revived with the development of semiclassical methods in quantum field theory, related to Euclidean classical solutions, the *instantons* [1, 2], (see e.g. good review [5]).

Although many questions concerning ensemble of instantons in QCD remain unclear, it was recently demonstrated that the instanton-induced effects are more important than previously believed, dominating even such quantities as baryonic masses, splittings and wave functions [3]. Evaluation of the next-order corrections, going beyond the standard semiclassical theory, are of great interest. As it is technically difficult to do such calculations for the gauge theory, one may naturally start with the quantum mechanical example. Note also, that some of the higher-dimensional problems can be directly connected to the double-well considered here.

This problem was first addressed in ref. [6]. Starting with the anharmonic oscillator, it was

shown how to apply Feynman diagrams in quantum mechanics for a few simple problems[*], and then the two-loop diagrams in the instanton background were evaluated. The nontrivial point here is the new type of diagram, coming from Jacobian of the collective mode.

Later an error in the Green's function used was noticed by Š.Olejnik [7], who corrected it and also made a new calculation of the instanton density, However, his results strongly disagree not only with those in [6], but also with numerical simulations of path integrals [8] and (which was discovered later) with the *exact* result[†] obtained by Zinn-Justin [9]. One may suspect either some fundamental problem of the method used; or a simple error. As we show in this letter, the method is fine and the error is found.

Let us introduce our notations (same as in [5]). A particle of mass m=1 moves in the potential

$$V_{inst} = \lambda(x^2 - \eta^2)^2 \tag{1}$$

The instanton solution is $X_{inst}(t) = \eta tanh(\frac{1}{2}\omega(t - t_c))$ where t_c is the instanton center and $\omega^2 = 8\lambda\eta^2$. Setting $\omega = 1$, and shifting one minimum to the origin one gets the anharmonic oscillator potential $V = \frac{1}{2}x^2 - \sqrt{2\lambda}x^3 + \lambda x^4$ with one small dimensionless parameter λ, to be used for the expansion. The classical action of the instanton solution $S_0 = \frac{1}{12\lambda}$ is therefore large, and we are looking for the corrections to the wave functions of the first and second state (i=1,2), the ground state energy and the splitting in the following form

$$\Psi_i(\eta) = \Psi_i^0(\eta)(1 + A/S_0 + \cdots) \tag{2}$$

$$E_0 = E_0^0(1 + B/S_0 + \cdots) \tag{3}$$

$$\delta E = \Delta E(1 + C/S_0 + \cdots) \tag{4}$$

Here $E_0^0 = \frac{\omega}{2}$ and the (one-loop) semiclassical result for the splitting is $\Delta E = 2\omega\sqrt{6S_0/\pi}e^{-S_0}$.

[*]This part can be recommended as methodical examples in quantum field theory courses: here Feynman diagrams appear without singularities of divergences, and the results can be easily checked by standard means.

[†]Zinn-Justin work is based on solution of Schroedinger equation in a form of series expansion: thus it certainly cannot be generalized to the field theory context we are mainly interested in.

Up to exponentially small corrections $\Psi_0(\pm\eta)$ and $\Psi_1(\pm\eta)$ are the symmetric and antisymmetric linear combinations of the ground state wavefunction of the perturbed harmonic oscillator, and therefore $A_1 = A_2 = A$.

The transition amplitude for $1/\Delta E >> \tau >> 1$ can therefore be written as

$$\langle -\eta \mid e^{-H\tau} \mid \eta \rangle \longrightarrow \mid \Psi_0^0(\eta) \mid^2 (1 + 2A/S_0 + \cdots) \times$$

$$\times e^{-\omega/2(1+B/S_0+\cdots)\tau} \Delta E(1 + C/S_0 + \cdots)\tau \tag{5}$$

Normalizing it to the transition amplitude $\langle \eta \mid exp(-H\tau) \mid \eta \rangle$ for the corresponding anharmonic oscillator, one can get rid of unwanted corrections A and B. In the Feynman path integral language, this ratio can be calculated by computing the diagrams in the instanton background (with the instanton-based vertices and the instanton-based Green's function), and then by subtracting the diagrams for the anharmonic oscillator (see [6] for details).

The instanton-based Green's function should satisfy the standard differential equation and also be orthogonal to the zero mode. The result [7] is

$$G(x,y) = G_0(x,y)[2-xy+1/4 \mid x-y \mid (11-3xy)+(x-y)^2]+3/8(1-x^2)(1-y^2)[\ln(G_0(x,y))-11/3]$$

$$\tag{6}$$

where $x = \tanh(t/2), y = \tanh(t'/2)$ and

$$G_0(x,y) = \frac{1- \mid x - y \mid -xy}{1+ \mid x - y \mid -xy} \tag{7}$$

The correction C we are interested in can be written as four diagrams for the instanton correction [6] $C = a_1 + b_{11} + b_{12} + c_1$ as shown in the Figure. Three first terms appear also for the anharmonic oscillator and should be subtracted, the last one comes from Jacobian of the zero mode and has no analogs in this case. Their analytic expressions are

$$a_1 = -3\beta \int_{-\infty}^{\infty} dt (G^2(t,t) - G_0^2(t,t))$$

$$= -\frac{97}{1680} \tag{8}$$

$$b_{11} = 3\alpha^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt dt' (\tanh(t/2)\tanh(t'/2)G^3(t,t') - G_0^3(t,t'))$$

$$= -\frac{53}{1260} \tag{9}$$

$$b_{12} = \frac{9}{2}\alpha^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt dt' (\tanh(t/2)\tanh(t'/2)G(t,t)G(t,t')G(t',t')$$

$$-G_0(t,t)G_0(t,t')G_0(t',t'))$$

$$= -39/560 \tag{10}$$

$$c_1 = -9\beta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt dt' \frac{\tanh(t/2)}{cosh^2(t/2)} \tanh(t'/2)G(t,t')G(t',t')$$

$$= -49/60 \tag{11}$$

from which follows $C = -71/72$.

This is exactly the result of J. Zinn-Justin in [9] ‡ . For comparison, Alejnikov and Shuryak [6] and Shuryak [8] got $C \approx -1$, and Olejnik [7] had $C \approx 0$.

The findings that (i) the correction has a coefficient C=O(1) and (ii) it is *negative* are important: they demonstrate that (i) the usual (one-loop) semiclassical formula does become inaccurate for not-so-large barriers ($S \sim 1$), and (ii) that it *overestimates* the tunneling probability. The magnitude of the corresponding coefficient in gauge theories remains unknown, although this question is of great practical importance.

Let us now switch to our second subject, the wave function correction (denoted A in (5)). Although it is much simpler than the problem discussed above (and has no dramatic history), it has a potential trap, which was not discussed for our knowledge and is worth mentioning.

Now, one has the general anharmonic potential $V_{anhosc}(x) = x^2/2 + \alpha x^3 + \beta x^4$ and evaluates

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‡The error made in ref.[7] was found to be related with diagrams b_{11}, b_{12}, in which the time-dependent triple vertex those were erroneously included for the subtracted anharmonic oscillator part as well.

the transition amplitude $\langle x = 0 \mid e^{-H_{anhosc}\tau} \mid x = 0 \rangle$. For τ going to infinity it gives terms proportional to τ which are the energy corrections and constant terms which are the wavefunction corrections.

Two different Green's functions can be used, which satisfy both the same differential equation but vanish at different boundaries. The infinite boundary Green's function is simply $G_{infinite}(t, t') = 1/2e^{-|t-t'|}$ while the finite one is

$$G_{finite}(t, t') = \sinh(\tau/2 - t)\sinh(t' + \tau/2)/\sinh(\tau) \quad (t \geq t')$$

$$= \sinh(\tau/2 - t')\sinh(t + \tau/2)/\sinh(\tau) \quad (t' \geq t) \tag{12}$$

Both give the same result for the energy shift, but *different* ones for the wave function correction. Up to the first order in β and α^2

$$A_{infinite} = -(29/96)\alpha^2; \quad A_{finite} = (9/16)\beta - (167/96)\alpha^2 \tag{13}$$

Comparison of the two results to the corrections obtained by old-fashioned perturbation theory shows that A_{finite} is correct[§]. The lesson is as follows: unlike the energy correction (which corresponds to infinitely long paths), the wave function is a kind of a boundary effect, and therefore proper restrictions for the quantum paths (and Green's functions) should not be ignored.

In conclusion, we have calculated the tunneling amplitude (equal to the energy splitting or the instanton density) up to two-loops, using Feynman diagrams. Two important elements of the calculation, making it somewhat less standard, is the *non-trivial background field* and *new effective vertices*, which come the zero-mode Jacobian rather than from the action. Our results agree exactly with ref. [9], based on a different method, and with other works inside the accuracy. In ref.[7] the error is found.

[§] Note that this observation does not invalidate the calculations above for the constant C: because the wave function corrections are the same for both the instanton and the anharmonic oscillator, up to exponentially small terms the constants A,B are cancelled whatever Green's function is used.

Extensions of this work can be (i) three and more loop corrections for quantum mechanical examples, and, more important (ii) possible two-loop calculations for gauge theories, in essentially the same setting. Such calculation would be of great interest, because the problem of large-size (or small action) instantons in QCD remains unsolved.

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$$(-3\beta) \qquad a_1 \qquad\qquad (3\alpha^2) \qquad b_{11} \qquad\qquad (\tfrac{9}{2}\alpha^2) \qquad b_{12}$$

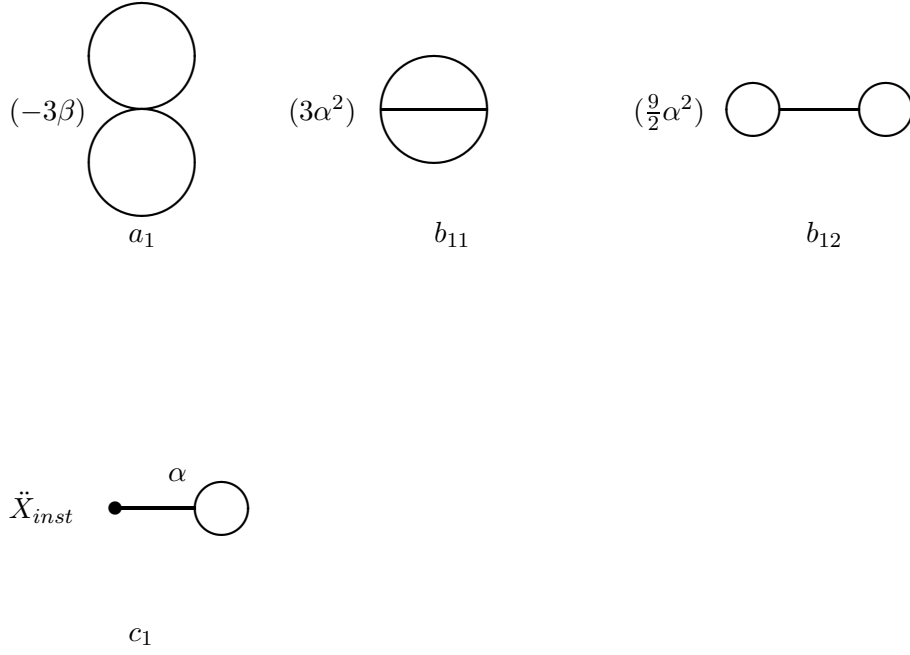$$\ddot{X}_{inst} \qquad \overset{\alpha}{\bullet\!\!-\!\!\bigcirc} \qquad c_1$$

Fig. 1.

Caption

Feynman diagrams for the two-loop correction to the energy splitting C. Here the triple and quartic coupling constants are $\alpha = -\sqrt{2\lambda} + 4X_{inst}(t)$, $\beta = \lambda$ for the instanton and $\alpha = -\sqrt{2\lambda}$, $\beta = \lambda$ for subtracted anharmonic oscillator. The indicated numbers of combinatorial origin should be added to ordinary Feynman rules, X_{inst} in the last diagram stands for the classical instanton path, dots are time derivatives.